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Transitivity of preferences: when does it matter?

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Abstract

We define the empirical conditions on prices and incomes under which transitivity of preferences has specific testable implications. In particular, we set out necessary and sufficient requirements for budget sets under which consumption choices can violate SARP (Strong Axiom of Revealed Preferences) but not WARP (Weak Axiom of Revealed Preferences). As SARP extends WARP by additionally imposing transitive preferences, this effectively defines the conditions under which transitivity is separately testable. Our findings have considerable practical relevance, as transitivity conditions are known to substantially aggravate the computational burden of empirical revealed preference analysis. Our characterization takes the form of triangular conditions that must hold for all three-element subsets of normalized prices, and which are easy to verify in practice. We demonstrate their practical use through two short empirical applications.

JEL Classification: C14, DO1, D11, D12.

Keywords: revealed preferences, WARP, SARP, transitive preferences, testable implications.

1 Introduction

Since Tversky (1969)'s seminal paper on intransitivity of preferences, the realism of transitive preferences has become a popular research topic in both psychology and (behavioral) economics (see, e.g., Regenwetter, Dana, and Davis-Stober (2011) for an overview). Nonetheless, in standard demand analysis transitivity of preferences is usually an obvious artefact of the consumer's optimization model. Importantly, however, in practical applications this assumption of transitive preferences can substantially aggravate the empirical analysis (see Section 2 for concrete examples). In this note, we define the empirical conditions on prices and incomes that characterize the empirical bite of transitivity. These conditions are necessary and sufficient for transitivity of preferences to have no specific

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testable implications. In other words, if (and only if) the conditions are met, then dropping transitivity will lead to exactly the same empirical conclusions. As a direct implication, we also characterize the demand data sets (i.e. prices and incomes) that allow one to meaningfully investigate transitivity of preferences.

A notable feature of our conditions is that they can be directly applied to the observed prices and incomes, without confounding the analysis by functional misspecification. In particular, we adopt a revealed preference approach that is intrinsically nonparametric, as it does not require a functional specification of consumers' utilities. This revealed preference approach was initiated by Samuelson (1938), who introduced the Weak Axiom of Revealed Preference (WARP) as a basic consistency requirement on consumption behaviour: if a consumer chooses a first bundle over a second one in a particular choice situation (characterized by a linear budget constraint), then (s)he cannot choose this second bundle over the first one in a different choice situation. Samuelson has also shown that WARP characterizes negativity of compensated demand effects. However, WARP imposes only a necessary condition for utility maximization, and does not exhaust all behavioral implications of the neoclassical consumer model.

More specifically, Houthakker (1950) has shown that a consumer behaves consistent with utility maximization if and only if it satisfies the Strong Axiom of Revealed Preference (SARP). Essentially, SARP extends WARP by additionally requiring transitivity of preferences. This makes it interesting to characterize the requirements for budget sets (i.e. prices and incomes) under which consumer choices can violate SARP but not WARP. Basically, this defines the empirical conditions under which transitivity of preferences has specific testable implications.

The question whether, and under what conditions, WARP and SARP are empirically distinguishable has attracted considerable attention in the literature on revealed preference theory. A first classic result is due to Rose (1958), who showed that WARP is equivalent to SARP when there are only two goods. Shortly afterwards, Gale (1960) proved, by counterexample, that WARP and SARP may differ in settings with more than two goods. Since then, various authors have presented further clarifications and extensions of Gale's basic result (see, e.g., Shafer (1977); Peters and Wakker (1994); Heufer (2014)). In a related vein, Uzawa (1960) showed that, if a demand function satisfies WARP together with some regularity condition, then it also satisfies SARP. However, Bossert (1993) put this result into perspective by demonstrating that, for continuous demand functions, Uzawa's regularity condition alone already implies SARP.

Previous studies typically exemplified the distinction between WARP and SARP by constructing hypothetical datasets (containing prices, incomes and consumption quantities) that satisfy WARP but violate SARP. Such datasets, however, might never be encountered in reality. In this sense, it leaves open the question whether the possibility to distinguish SARP from WARP is merely a theoretical curiosity or also an empirical regularity. In addition, the datasets that are constructed do not define general conditions on budget sets (i.e. prices and incomes, without quantities) under which SARP and WARP are empirically equivalent (or, conversely, transitivity is separately testable).

In this note, we provide necessary and sufficient requirements for budget sets under which WARP is equivalent to SARP, i.e. any possible configuration of quantity choices satisfies WARP if and only if it satisfies SARP. The fact that our conditions are defined in terms of budget sets, without requiring quantity information, is particularly convenient from a practical point of view. It makes it possible to check on the basis of given prices and incomes whether it suffices to (only) check WARP (instead of SARP) to verify consistency with utility maximization. Conversely, it characterizes the budget conditions under which transitivity of preferences has separate empirical implications and, thus, for which transitivity restrictions can potentially add value to the analysis.

A main practical motivation for our theoretical analysis relates to the computational issues associated with the verification of revealed preference axioms. In particular, whether or not transitivity concerns are taken into account (i.e. SARP-based versus WARP-based) bears heavily on the computational burden of empirical revealed preference analysis (see Echenique, Lee, and Shum (2011); Kitamura and Stoye (2013); Blundell, Browning, Cherchye, Crawford, De Rock, and Vermeulen (2015) for some recent examples). In general, dropping transitivity can considerably alleviate the computational efforts associated with the analysis. This consideration becomes all the more important given that increasingly large consumption datasets are becoming available. Attractively, our conditions are easy to verify in practice, even for such large datasets.

Section 2 provides a more detailed discussion of the practical relevance of our conditions for SARP to be equivalent to WARP. Section 3 first introduces some notation and basic definitions, and subsequently presents our main result as a generalization of Rose (1958)'s original result. Section 4 introduces two interesting extensions of this main result. Specifically, it establishes a connection with Hicksian aggregation, and it defines a dual formulation (and interpretation) of our conditions. Section 5 shows the practical use of our theoretical findings through two short applications. Finally, Section 6 concludes.

2 Practical relevance

Dropping transitivity conditions (i.e. WARP-based instead of SARP-based) can considerably mitigate the computational difficulties in empirical revealed preference analysis. In what follows, we will illustrate this by discussing alternative settings in which computational concerns are important, and which have received considerable attention in the applied literature. This will directly motivate the empirical relevance of the theoretical results that we present in the next sections.

As indicated above, revealed preference axioms (like WARP and SARP) can be directly applied to the data. They allow us to assess and compare the empirical performance of alternative models of consumer behaviour in a fully nonparametric manner (see Blow, Browning, and Crawford (2008); Cherchye, De Rock, and Vermeulen (2009); Blow, Browning, and Crawford (2014); Adams, Cherchye, De Rock, and Verriest (2014); Demuynck and Seel (2014); Adams, Blundell, Browning, and Crawford (2015) for some recent examples). In this respect, an important consideration concerns the "power" of the revealed preference axioms that are subject to evaluation. In words, we can define power as the probability of detecting random behaviour. For observed prices and incomes, it is computed by drawing a large number of randomly generated quantity choices that exhaust the available budgets (see Bronars (1987)). The power of a revealed preference axiom is then obtained as one minus the fraction of such random datasets that satisfy the axiom. As such, computing power requires the verification of WARP or SARP on a very large number of datasets. In general, calculating the power of SARP requires substantially more computational effort than calculating the power of WARP. Clearly, this can be avoided for budget sets under which SARP is known to be empirically equivalent to WARP.

A closely related setting pertains to the computation of the Houtman-Maks index (Houtman and Maks, 1985). In cases where the full dataset of consumer choices is found to be inconsistent with SARP, this index equals the size of the largest subset of data that does satisfy SARP. The computation of this Houtman-Maks index is known to be computationally difficult (technically, it is an "NP-hard" problem).[1] When WARP is equivalent to SARP, the computation of the Houtman-Maks index is equivalent to the vertex set cover problem. Although this is again a complex (i.e. NP-hard) problem, there do exist very quick algorithms that give the right solution for most instances (see Gross and Kaiser (1996)).

A next relevant context concerns the so-called "stochastic" axioms of revealed preference, which form the population analogues of the more standard revealed preference axioms such as WARP and SARP (see McFadden (2005) for an overview). In a stochastic revealed preference setting, the verification of WARP is relatively easy from a computational point of view (see, e.g., Hoderlein and Stoye (2014) and Cosaert and Demuynck (2014)), while the verification of SARP is known to be difficult (i.e. NP-hard; see, e.g., Kitamura and Stoye (2013)). As a direct implication, the knowledge that WARP is empirically equivalent to SARP can have a huge impact on the computation time.

All the above examples pertain to settings in which revealed preference axioms are directly applied to the actual choice data (i.e. quantities and prices for observed decision situations). Recently, there is also an emerging strand of literature that focuses on combining revealed preference axioms with estimated consumer demand functions, so aiming at a more powerful empirical analysis. For instance, Blundell, Browning, and Crawford (2003, 2008) showed that combining estimated Engel curves with revealed preference axioms can obtain tight bounds on cost of living indices and demand responses.[2] It turns out that the algorithms that are needed for this purpose are profoundly more elaborate when considering SARP instead of WARP (see Blundell, Browning, Cherchye, Crawford, De Rock, and Vermeulen (2015)).

As a concluding remark, we indicate that our results may also be relevant from a noncomputational point of view. They can equally be useful for the design of experiments

[1]Recently, Crawford and Pendakur (2013) proposed a way to deal with unobserved heterogeneity in empirical revealed preference analysis by using concepts that are formally close to this Houtman-Maks index. The practical operationalization of these authors' proposal suffers from the same computational complexity problem.

[2]Cherchye, De Rock, Lewbel, and Vermeulen (2015) provide similar results regarding sharing rule identification in the context of collective models of household consumption. These authors also point out the computational issues that relate to explicitly integrating transitivity restrictions in practical applications.

that aim at testing revealed preference axioms in a laboratory setting (in the tradition of Tversky (1969)). For example, one might be interested in separately testing transitivity of preferences. This requires budget sets for which SARP is not equivalent to WARP, which we characterize in our following analysis.

3 When WARP equals SARP

We assume a consumer who composes bundles of m goods for n budget sets. This defines a dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ with price vectors $\mathbf{p}_t \in \mathbb{R}^m_{++}$ and quantity vectors $\mathbf{q}_t \in \mathbb{R}^m_{+}$. To facilitate our further discussion, we summarize the budget conditions in terms of normalized prices, which implies total expenditures $\mathbf{p}_t\mathbf{q}_t = 1$ for all observations $t = 1, \ldots, n$. We can now define the basic revealed preference concept.

Definition 1. *The bundle $\mathbf{q}_t$ at observation t is* ***revealed preferred*** *to the bundle $\mathbf{q}_v$ at observation v if $\mathbf{p}_t\mathbf{q}_t(= 1) \geq \mathbf{p}_t\mathbf{q}_v$. We denote this as $\mathbf{q}_t R \mathbf{q}_v$.*

In words, $\mathbf{q}_t$ is revealed preferred to $\mathbf{q}_v$ if $\mathbf{q}_v$ was cheaper than $\mathbf{q}_t$ at the prices observed at t. Then, we have the following definitions of WARP and SARP.

Definition 2. *A dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ violates* ***WARP*** *if R has a cycle of length 2, i.e. $\mathbf{q}_t R \mathbf{q}_v R \mathbf{q}_t$ and $\mathbf{q}_t \neq \mathbf{q}_v$.*

Definition 3. *A dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ violates* ***SARP*** *if R has a cycle, i.e. $\mathbf{q}_t$ R $\mathbf{q}_v R$ $\mathbf{q}_s$ $\ldots$ R $\mathbf{q}_k$ R $\mathbf{q}_t$ for some sequence of observations $t, v, s, \ldots, k$ and not all bundles $\mathbf{q}_t, \ldots, \mathbf{q}_k$ are identical.*

It is clear from the definitions that SARP consistency implies WARP consistency. We are interested in the reverse relationship: under which conditions does WARP imply SARP? Given our specific research question, we consider settings in which the empirical analyst does not necessarily observe the quantity choices, but only the normalized prices (i.e. budget sets). For the given normalized prices, we are interested in the possibility that there exist corresponding quantity bundles that imply a SARP or WARP violation. To this end, we use the following definition.

Definition 4. *A set of prices $\{\mathbf{p}_t|t = 1, \ldots, n\}$ is said to be* ***WARP-reducible*** *if, for any set of quantities $\{\mathbf{q}_t|t = 1, \ldots, n\}$ for which $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ violates SARP, we also have that $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ violates WARP.*

To set the stage, we first repeat Rose (1958)'s classical result, which says that WARP is always equivalent to SARP if the number of goods equals two (i.e. $m = 2$). We phrase this result in terms of the terminology that we introduced above.

Proposition 1. *If there are only two goods (i.e. $m = 2$), then any set of prices $\{\mathbf{p}_t|t = 1, \ldots, n\}$ is WARP-reducible.*

Our main result will provide a generalization of Proposition 1. It makes use of the concept of a triangular configuration.

Definition 5. *A set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is a* ***triangular configuration*** *if, for any three price vectors* $\mathbf{p}_t, \mathbf{p}_v$ *and* $\mathbf{p}_k$ *(with* $t, v, k \in \{1, \ldots, n\}$*), there exists a number* $\lambda \in [0, 1]$ *and a permutation* $\sigma : \{t, v, k\} \to \{t, v, k\}$ *such that the following condition holds:*

$$\mathbf{p}_{\sigma(t)} \leq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)} \text{ or } \mathbf{p}_{\sigma(t)} \geq \lambda \mathbf{p}_{\sigma(v)} + (1 - \lambda)\mathbf{p}_{\sigma(k)}.$$

Note that the inequalities in this definition are vector inequalities. As such, Definition 5 states that, for any three vectors, we need that there is a convex combination of two of the three prices that is either smaller or larger than the third price vector. Checking whether a set of prices is a triangular configuration merely requires verifying the linear inequalities in Definition 5 for any possible combination of three prices. Clearly, this is easy to do in practice, even if the number of observations (i.e. n) gets large.

We can show that the triangular conditions in Definition 5 are necessary and sufficient for WARP and SARP to be equivalent. The proof of Proposition 2 is presented in Appendix A.

Proposition 2. *A set of prices* $\{\mathbf{p}_t | t = 1, \ldots, n\}$ *is WARP-reducible if and only if it is a triangular configuration.*

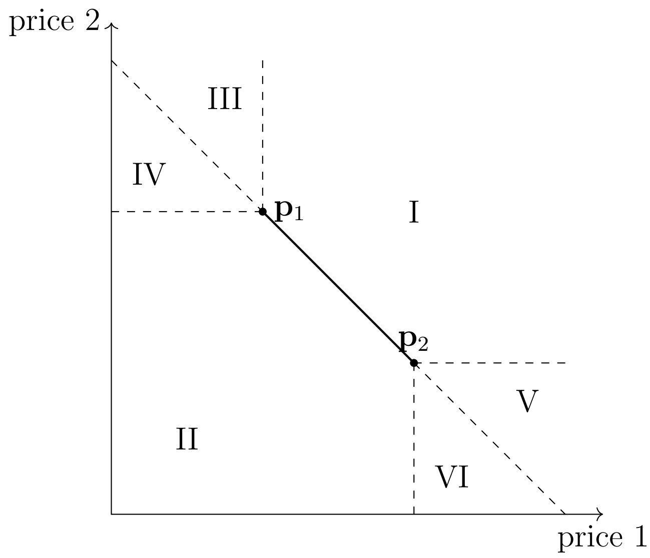
This results generalizes Rose's result in Proposition 1. In particular, one can verify that, if the number of goods is equal to two, then any set of prices is a triangular configuration. To see this, consider three normalized price vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for two goods (i.e. $m = 2$). Obviously, if $\mathbf{p}_1 \geq \mathbf{p}_2$ or $\mathbf{p}_2 \geq \mathbf{p}_1$, we have that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a triangular configuration. Let us then consider the more interesting case where $\mathbf{p}_1$ and $\mathbf{p}_2$ are not ordered, which we illustrate in Figure 1. The price vector $\mathbf{p}_3$ can fall into six regions, which are numbered I to VI. For any of these six possible scenarios, the triangular condition in Definition 5 is met. To see this, we first consider the case where $\mathbf{p}_3$ lies in region I. In that case, $\mathbf{p}_3$ is obviously larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Similarly, if $\mathbf{p}_3$ lies in region II, it is smaller than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$. Next, if $\mathbf{p}_3$ lies in region III, then $\mathbf{p}_1$ is smaller than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ and, conversely, $\mathbf{p}_1$ is larger than a convex combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ if $\mathbf{p}_3$ lies in region IV. Finally, if $\mathbf{p}_3$ lies in region V, there is a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$ that dominates $\mathbf{p}_2$ and, if $\mathbf{p}_3$ lies in region VI, then $\mathbf{p}_2$ is larger than a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$. We conclude that any possible set of prices $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is WARP-reducible.

Example 1 provides some further intuition for the result in Proposition 2. In this example, we focus on cycles of length 3, and show that the triangular configuration implies that each SARP violation of length 3 must contain a WARP violation.

Example 1. *Consider a set of three prices* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *that is a triangular configuration. Without loss of generality, we may assume that* P *is a triangular configuration because one of the following two inequalities holds:* $\mathbf{p}_1 \leq \lambda \mathbf{p}_2 + (1 - \lambda)\mathbf{p}_3$ *or* $\mathbf{p}_1 \geq \lambda \mathbf{p}_2 + (1 - \lambda)\mathbf{p}_3$ *for some* $\lambda \in [0, 1]$.

Let us first consider $\mathbf{p}_1 \leq \lambda \mathbf{p}_2 + (1 - \lambda)\mathbf{p}_3$*. Assume that there exists a SARP violation with a cycle of length 3. With three observations, there are only two possibilities for cycles*

Figure 1: The triangular condition in a two goods setting



of length 3: $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$ *or* $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$. *If* $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$, *then it must be that*

$$1 = \mathbf{p}_2 \mathbf{q}_2 \geq \mathbf{p}_2 \mathbf{q}_3 \text{ and } 1 = \mathbf{p}_3 \mathbf{q}_3.$$

Together with our triangular inequality this implies that

$$1 \geq (\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3)\mathbf{q}_3 \geq \mathbf{p}_1 \mathbf{q}_3.$$

As such, we can conclude that $\mathbf{q}_1 R \mathbf{q}_3$, *which gives* $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_1$, *i.e. a violation of WARP. A similar reasoning holds for the second possibility (i.e.* $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$*), which shows that in this first case each violation of SARP implies a WARP violation.*

For the second case, $\mathbf{p}_1 \geq \lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3$, *we must consider the same two possible SARP violations. The reasoning is now slightly different. In particular, let us assume that there is no violation of WARP. For the SARP violation* $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 R \mathbf{q}_1$ *this requires* $1 < \mathbf{p}_3 \mathbf{q}_2$ *(i.e. not* $\mathbf{q}_3 R \mathbf{q}_2$ *). Since* $1 = \mathbf{p}_2 \mathbf{q}_2$, *we obtain that, if* $\lambda < 1$,

$$1 < (\lambda \mathbf{p}_2 + (1-\lambda)\mathbf{p}_3)\mathbf{q}_2 \leq \mathbf{p}_1 \mathbf{q}_2.$$

This clearly contradicts $\mathbf{q}_1 R \mathbf{q}_2$ *(i.e.* $1 \geq \mathbf{p}_1 \mathbf{q}_2$*). If* $\lambda = 1$, *we have* $\mathbf{p}_1 \geq \mathbf{p}_2$ *and thus*

$$1 = \mathbf{p}_1 \mathbf{q}_1 \geq \mathbf{p}_2 \mathbf{q}_1.$$

This again yields a contradiction, as it implies the WARP violation $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_1$. *A similar reasoning holds for the second possibility (i.e.* $\mathbf{q}_1 R \mathbf{q}_3 R \mathbf{q}_2 R \mathbf{q}_1$*), which shows that also for this case any SARP violation implies a WARP violation.*

4 Extensions

Before proceeding to our empirical illustrations, we present two interesting extensions of our main result in Proposition 2. These two extensions provide some further intuition for our characterization of WARP-reducibility, by establishing a relation with quantity and price aggregation. The first extension shows a close connection between our characterization and Hicksian aggregation. The second extension derives a dual formulation of the triangular conditions in Proposition 2, which defines conditions on quantities for WARP-reducibility.

Hicksian aggregation. We can interpret a special instance of the triangular conditions in Proposition 2 as a generalization of Hicksian quantity aggregation. To formalize the argument, we first note that a special case of triangular configuration occurs if, for all triples of prices, the triangle inequalities in Definition 5 in fact become an equality, i.e.

$$\mathbf{p}_{\sigma(t)} = \lambda \mathbf{p}_{\sigma(v)} + (1-\lambda)\mathbf{p}_{\sigma(k)}.$$

This implies that all prices lie in a common 2-dimensional plane, which effectively means that there exist vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m_+$ and observation-specific numbers $\alpha_t, \beta_t > 0$ such that, for all $t \in \{1, \ldots, n\}$, we have $\mathbf{p}_t = \alpha_t \mathbf{v} + \beta_t \mathbf{w}$.

Interestingly, this actually allows us to rewrite the original setting as a setting with two Hicksian aggregates. Hicksian aggregation requires that all prices in a subset of goods change proportionally to some common price vector (i.e. $\mathbf{p}_t = \alpha_t \mathbf{v}$ for all t, with $\mathbf{v} \in \mathbb{R}^m_+$ and scalar $\alpha_t > 0$). In our case, we can, for any bundle $\mathbf{q}_t$, construct two new "aggregate quantities" $z_{t,1} = \mathbf{v}\mathbf{q}_t$ and $z_{t,2} = \mathbf{w}\mathbf{q}_t$, to define the quantity bundle $\mathbf{z}_t = [z_{t,1}, z_{t,2}]$. Correspondingly, we can construct new "price vectors" $\mathbf{r}_t = [\alpha_t, \beta_t]$. Then, for any two observations t and v, we have $\mathbf{q}_t R \mathbf{q}_v$ if and only if

$$1 \geq \mathbf{p}_t \mathbf{q}_v = (\alpha_t \mathbf{v} + \beta_t \mathbf{w})\mathbf{q}_v = \alpha_t \mathbf{v}\mathbf{q}_v + \beta_t \mathbf{w}\mathbf{q}_v = \mathbf{r}_t \mathbf{z}_v.$$

In other words, we obtain $\mathbf{q}_t R \mathbf{q}_v$ for the dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ if and only if $\mathbf{z}_t R \mathbf{z}_v$ for the dataset $\{(\mathbf{r}_t, \mathbf{z}_t) | t = 1, \ldots, n\}$. This implies that the dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ will violate SARP (resp. WARP) if and only if the dataset $\{(\mathbf{r}_t, \mathbf{z}_t) | t = 1, \ldots, n\}$ violates SARP (resp. WARP). Moreover, the dataset $\{(\mathbf{r}_t, \mathbf{z}_t) | t = 1, \ldots, n\}$ only contains two goods, so Proposition 1 implies that WARP is equivalent to SARP, and this equivalence carries over to the dataset $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$. Basically, this defines the possibility to construct two Hicksian aggregates as a (sufficient) condition for WARP to be equivalent to SARP.

Dual formulation. We can also define dual analogues of our notions of WARP-reducibility for sets of quantities (by interchanging the roles of prices and quantities in Definition 4). Similar to before, a set of quantities will be WARP-reducible if and only if it satisfies a set of triangular conditions. In this case, these conditions require, for any three quantity vectors $\mathbf{q}_t, \mathbf{q}_v$ and $\mathbf{q}_k$, that there must exist a number $\lambda \in [0,1]$ and a permutation

$\sigma : \{t, v, k\} \to \{t, v, k\}$, such that the following condition holds:

$$\mathbf{q}_{\sigma(t)} \leq \lambda \mathbf{q}_{\sigma(v)} + (1 - \lambda)\mathbf{q}_{\sigma(k)} \text{ or } \mathbf{q}_{\sigma(t)} \geq \lambda \mathbf{q}_{\sigma(v)} + (1 - \lambda)\mathbf{q}_{\sigma(k)}.$$

Following a similar logic as above, a special instance of these triangular conditions (with an equality instead of two inequalities) can be reinterpreted in terms of a dual version of Hicksian aggregation. In this case, a (sufficient) condition for WARP-reducibility is that it is possible to additively decompose quantities into two (implicit) vectors of quantities that vary equiproportionately (defined over all possible prices). Like before, this in turn allows us to represent the original setting as a two-goods setting, which is now defined in terms of two "aggregate prices".

Intuitively, this dimension reduction is somewhat related to the notion of demand system rank. See Lewbel (1991) for a general discussion of this rank concept, and Blundell and Robin (2000) for an interpretation in terms of grouping goods with latent separability. In a sense, we here provide a revealed preference version of such a dimension reduction. More generally, our dual formulation presented above may define a fruitful starting point to provide a revealed preference characterization of the rank of a demand system.

5 Empirical applications

To show the practical relevance of our triangular conditions, we present empirical applications that make use of two different types of datasets that have been the subject of empirical revealed preference analysis in recent studies. They will illustrate alternative possible uses of our characterization in Proposition 2.

Consumer survey data. Our first application uses the data from the British Family Expenditure Survey (FES) that have been analysed by Blundell et al. (2003, 2008, 2015). As indicated in Section 2, these authors developed methods to combine Engel curves with revealed preference axioms to obtain tight bounds on cost of living indices and demand responses. These methods become substantially more elaborate when considering SARP instead of WARP. This makes it directly relevant to check whether WARP and SARP are equivalent for the budget sets taken up in the analysis.

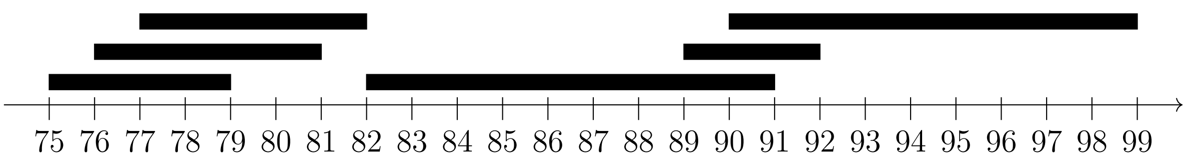
More specifically, the dataset contains 25 yearly observations (1975 to 1999) for three product categories (food, other nondurables and services). As in the original studies we focus on mean income. When checking our triangular conditions for all triples of (normalized) prices, we conclude that 2.39% of these triples violate these conditions. This indicates that WARP and SARP are not fully equivalent for these data. However, for a fraction as low as 2.39%, it is also fair to conclude that the subset of prices that may induce differences between WARP and SARP is quite small.

As a further exercise, we identified the largest subset of the 25 observation years that does satisfy the triangular conditions in Proposition 2.[3] It turns out that this largest

[3]This subset can be identified by solving a simple integer programming problem (with binary integer variables). The program is available upon request.

triangular-consistent subset contains 17 observed budget sets. Putting it differently, if we drop 8 of the 25 original observations, we know that WARP-based and SARP-based analyses will obtain exactly the same conclusions.

Figure 2: Largest triangular consistent subperiods (FES)



In a last step, we redo the previous analysis but now we focus on continuous subperiods of the full period 1985-1999 that are consistent with our triangular conditions. This can provide guidance, for example, for breaking up the total set of observations into subsets, to subsequently conduct a separate WARP-based (or, equivalently, SARP-based) analysis for every other subset. The results of this exercise are reported in Figure 2. It turns out that the longest subperiods for which WARP and SARP are equivalent contain ten years (1982-1991 and 1990-1999). By contrast, the shortest continuous subperiod that satisfies our triangular conditions has only four years (1989-1992).

Household scanner data. Our second application uses the data from the Stanford Basket Dataset, which is a detailed household-level scanner panel dataset. It contains grocery expenditure data for 494 households from four stores in an urban area of a large U.S. mid-western city, between June 1991 and June 1993. Echenique, Lee, and Shum (2011) have used this dataset in a SARP-based revealed preference analysis to compute their so-called Money Pump Index. To exactly compute this index, they have to consider all possible SARP violations in a given data set. As explicitly indicated by the authors, this is a huge computational task and, for this reason, they only approximate their index by focusing on SARP violations that involve at most four observations.

We have a separate dataset for each of the 494 households, which contain on average 24 observations per household on expenditures over a large number of commodities that cover 13 categories.[4] For each household dataset, we check whether the budget sets satisfy our triangular conditions. Like before, we compute the fraction of price triples that violate the condition, and we identify the largest subgroups of price observations that are triangular configurations.

Table 1 summarizes a first set of results. It reports on the distribution of violations (expressed as a fraction of all possible price triples) of our triangular conditions for our sample of 494 households. Interestingly, we find that none of the 494 household datasets satisfies our triangular conditions for all triples of prices (i.e. the minimum is positive), which means that WARP is nowhere fully equivalent to SARP. Further, we observe that the fraction of violations may differ considerably across households. It varies between only 1% and no less than 45%. Apparently, for some households there are quite many price

[4]The 13 categories are bacon, barbecue, butter, cereal, coffee, crackers, eggs, ice cream, nuts, analgesics, pizza, snacks and sugar.

Table 1: Results for Stanford Basket Dataset (original selection of goods)

	violations (as fraction of all possible triples)	largest subset (as fraction of all observations)
mean	0.0947	0.4950
min	0.0103	0.2308
25th percentile	0.0474	0.4348
median	0.0756	0.5000
75th percentile	0.1174	0.5600
max	0.4558	0.7895

observations that can imply a difference between WARP and SARP, whereas the opposite holds for other households. When looking at the actual consumption data, we observe that 11.7% of the household satisfy WARP or SARP. There are no households that violate SARP but satisfy WARP.

If we look at the size of the largest subsets of observations that satisfy our triangular conditions, Table 1 shows that, again, there is quite some variation across households. For example, for one household this largest subset contains no more than 23% of the observations, whereas for another household it contains as much as 79% of the observations. All this leads us to conclude that differences between WARP-based and SARP-based analyses may vary a lot depending on the household under consideration.

As a final exercise, we investigate whether and to what extent the stringency of our triangular conditions depends on the number of goods. For this we gradually reduce the dimension of the price vector by using a factorization procedure. The idea of this procedure is to reduce the number of goods, while keeping the revealed preference comparisons as close as possible to those for the original data. We provide more details of this factorization procedure in Appendix B.

Our results are given in Table 2, which has a similar interpretation as Table 1. In general, we find that the fraction of price triples that violate the triangular conditions ("viol." in Table 2) increases as the number of goods increases. In fact, if the number of goods is sufficiently small, we have that our triangular conditions are met for a substantial number of household datasets (e.g. the 25th percentile equals zero for $k = 3$). In these cases, WARP and SARP are equivalent. In a similar vein, the size of the largest subset consistent with the triangular conditions ("max. sub." in Table 2) generally decreases with the number of goods. These results confirm the intuition that the triangular conditions become more restrictive when the number of goods increases. As an implication, the empirical implications of WARP and SARP will usually differ more for higher numbers of goods.

Table 2: Results of Stanford Basked dataset for various numbers (k) of goods

	$k = 3$		$k = 5$		$k = 10$		$k = 15$	
	viol.	max. sub.	viol	max. sub.	viol.	max. sub.	viol.	max. sub.
mean	0.0115	0.8466	0.0397	0.7174	0.0852	0.6090	0.1046	0.5822
min	0	0.5000	0	0.3077	0	0.2500	0	0.2308
25th percentile	0	0.7308	0.0074	0.5909	0.0198	0.5000	0.0287	0.4583
median	0.0064	0.8462	0.0219	0.7083	0.0534	0.6000	0.0628	0.5769
75th percentile	0.0168	1	0.0519	0.8421	0.1105	0.7083	0.1409	0.6818
max	0.1331	1	0.3221	1	0.5627	1	0.6515	1.0000

6 Conclusion

We have presented triangular conditions for budget sets that are necessary and sufficient for WARP and SARP to be empirically equivalent. This defines the empirical conditions under which transitivity of preferences has separate testable implications. Conveniently, our triangular conditions are easy to check in practice. The conditions can be particularly relevant in settings where a SARP-based analysis requires substantially more computational effort than a WARP-based analysis. We also conducted two empirical applications that illustrate alternative possible uses of our conditions.

A Proof of Proposition 2

Before we give the proof of our main result, let us introduce some notation. For a finite set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$, consider the **convex hull** of P,

$$C(P) = \left\{ \mathbf{p} \in \mathbb{R}^m_{++} \middle| \mathbf{p} = \sum_{t=1}^{n} \alpha_t \mathbf{p}_t, \alpha_t \geq 0, \sum_{t=1}^{n} \alpha_t = 1 \right\},$$

and the **convex monotone hull** of P,

$$CM(P) = \left\{ \mathbf{p} \in \mathbb{R}^m_{++} \middle| \mathbf{p} \geq \sum_{t=1}^{n} \alpha_t \mathbf{p}_t, \alpha_t \geq 0, \sum_{t=1}^{n} \alpha_t = 1 \right\}.$$

The set $C(P)$ contains all prices that are a convex combination of the prices in P, while the set $CM(P)$ contains all prices that are at least as large as a convex combination of the prices in P. A price vector $\mathbf{p}_t$ is a called a **vertex** of $CM(P)$ if $\mathbf{p}_t \notin CM(P \setminus \{\mathbf{p}_t\})$. It is easy to verify that every element in $CM(P)$ is larger than or equal to some convex combination of the vertices of $CM(P)$.

Consider a bundle $\mathbf{q} \in \mathbb{R}^m_+$ that satisfies $\mathbf{p}_t\mathbf{q} = 1$. Then, the set of vectors $\mathbf{p}$

$$H(\mathbf{q}) = \{\mathbf{p} | \mathbf{p}\mathbf{q} = 1\},$$

defines an $(m-1)-$dimensional hyperplane in the space $\mathbb{R}^m$. Of course, we have that $\mathbf{p}_t \in H(\mathbf{q})$. For a non-zero vector $\mathbf{q} \in \mathbb{R}^m_+$, the hyperplane $H(\mathbf{q})$ is said **to cut** the set C

if there are two vectors $\mathbf{p}, \mathbf{p}' \in C$ such that $1 \leq \mathbf{pq}$ and $1 \geq \mathbf{p}'\mathbf{q}$. If C is non-empty and monotone (i.e. if $\mathbf{p} \in C$ and $\mathbf{p}' \geq \mathbf{p}$, then $\mathbf{p}' \in C$), then we can always find a vector $\mathbf{p}$ that satisfies the first inequality. In this case, only the second inequality is relevant.

Finally, for a number j we write $\lfloor j \rfloor$ for $(j \mod n)$. We start by proving two lemmata.

Lemma 1. *Consider a set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$ and a non-zero consumption bundle $\mathbf{q}$ where $\mathbf{p}_t\mathbf{q} = 1$. If the hyperplane $H(\mathbf{q})$ cuts $CM(P) \setminus \{\mathbf{p}_t\}$, then there is a vertex $\mathbf{p}_v \in CM(P)$, distinct from $\mathbf{p}_t$, such that $1 \geq \mathbf{p}_v\mathbf{q}$.*

Proof. If $H(\mathbf{q})$ cuts $CM(P) \setminus \{\mathbf{p}_t\}$, then there is a vector $\mathbf{p} \in CM(P)$, with $\mathbf{p} \neq \mathbf{p}_t$, such that $1 \geq \mathbf{pq}$. From the definition of $CM(P)$ there must exist numbers $\alpha_j \geq 0$, with $\sum_j \alpha_j = 1$ and

$$1 \geq \mathbf{pq} \geq \left(\sum_{j=1}^{n} \alpha_j \mathbf{p}_j \right) \mathbf{q} = \sum_{j=1}^{n} \alpha_j \mathbf{p}_j \mathbf{q},$$

As mentioned above, without loss of generality, we may assume that all $\mathbf{p}_j$ corresponding to a strict positive α_j are vertices.

Let $J = \arg\min_j \mathbf{p}_j\mathbf{q}$ where j is restricted to those values with $\alpha_j > 0$. If there is a $j \in J$ with $\mathbf{p}_j \neq \mathbf{p}_t$, we obtain that $1 \geq \mathbf{p}_j\mathbf{q}$ what we needed to proof. On the other hand, if $J = \{t\}$, then (from $\mathbf{p}_t \neq \mathbf{p}$)

$$1 \geq \mathbf{pq} \geq \sum_{j=1}^{n} \alpha_j \mathbf{p}_j \mathbf{q} > \mathbf{p}_t \mathbf{q}.$$

This gives a contradiction with $\mathbf{p}_t\mathbf{q} = 1$. □

The following lemma is similar to Theorem 1 in Heufer (2014), but it is stated in terms of prices instead of quantities.

Lemma 2. *Let $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$ be a set of prices and let $\{\mathbf{q}_t | t = 1, \ldots, n\}$ be a set of distinct non-zero bundles such that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates SARP. Also assume that no strict subset of $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates SARP. Without loss of generality, assume that the SARP violation is given by $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 \ldots R \mathbf{q}_n R \mathbf{q}_1$ (i.e. $1 \geq \mathbf{p}_1\mathbf{q}_2, 1 \geq \mathbf{p}_2\mathbf{q}_3, \ldots, 1 \geq \mathbf{p}_n\mathbf{q}_1$). Then,*

1. *the prices in P are the vertices of the set $CM(P)$;*
2. *for all $\alpha \in]0, 1[$ and $j = 1, \ldots, n$ the vector $\alpha\mathbf{p}_j + (1-\alpha)\mathbf{p}_{\lfloor j+1 \rfloor}$ is not in the relative interior of $CM(P)$, i.e. there do not exist numbers $\alpha_t \geq 0, \sum_t \alpha_t = 1$ such that,*

$$\alpha \mathbf{p}_j + (1-\alpha)\mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t=1}^{n} \alpha_t \mathbf{p}_t.$$

Proof. Assume, towards a contradiction, that $\mathbf{p}_j$, with $j \in \{1, \ldots, n\}$, is not a vertex of $CM(P)$. Since $CM(P)$ is monotone and $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1 \rfloor}$, we know that the hyperplane $H(\mathbf{q}_{\lfloor j+1 \rfloor})$ cuts $CM(P)$. From Lemma 1 we therefore obtain that there exists some vertex

$\mathbf{p}_v$ of $CM(P)$, such that $1 \geq \mathbf{p}_v\mathbf{q}_{\lfloor j+1 \rfloor}$. As $\mathbf{p}_j$ is not a vertex, $\mathbf{p}_v \neq \mathbf{p}_j$. If $v < j < n$, we have that $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, v, j+1, \ldots, n\}$ violates SARP. If $v < j = n$, we have that $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, v\}$ violates SARP. Finally, if $v > j$, we obtain that $\{(\mathbf{p}_t, \mathbf{q}_t)|t = j+1, \ldots, v\}$ violates SARP. In all cases we thus obtain the desired contradiction as a strict subset of $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ violates SARP.

For the second part, assume, again towards a contradiction, that there exists an $\alpha \in]0, 1[$ such that $\mathbf{p}' = \alpha\mathbf{p}_j + (1-\alpha)\mathbf{p}_{\lfloor j+1 \rfloor}$ is in the relative interior of $CM(P)$. That is, there exist numbers $\alpha_t \geq 0$, with $\sum_{t=1}^n \alpha_t = 1$, such that

$$\alpha\mathbf{p}_j + (1-\alpha)\mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t=1}^n \alpha_t \mathbf{p}_t.$$

Observe that $\alpha_t > 0$ for at least one $t \notin \{j, \lfloor j+1 \rfloor\}$, since $\mathbf{p}_j$ and $\mathbf{p}_{\lfloor j+1 \rfloor}$ are both vertices of CM(P). Rewriting this inequality gives

$$(\alpha - \alpha_j)\mathbf{p}_j + (1 - \alpha - \alpha_{\lfloor j+1 \rfloor})\mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \alpha_t \mathbf{p}_t,$$

where α_j or $\alpha_{\lfloor j+1 \rfloor}$ are potentially equal to zero. Given that the right hand side is strictly positive, one of the terms $(\alpha - \alpha_j)$ or $(1 - \alpha - \alpha_{\lfloor j+1 \rfloor})$ should be strictly positive. If the first term is strictly positive and the second term is non-positive, then

$$\mathbf{p}_j > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \frac{\alpha_t}{\alpha - \alpha_j}\mathbf{p}_t + \frac{\alpha - 1 + \alpha_{\lfloor j+1 \rfloor}}{\alpha - \alpha_j}\mathbf{p}_{j+1}.$$

This shows that $\mathbf{p}_j$ is in $CM(P \setminus \{\mathbf{p}_j\})$, a contradiction with the first part of the lemma. Similarly, if the first term is non-positive and the second term strictly positive, then

$$\mathbf{p}_{\lfloor j+1 \rfloor} > \sum_{t \notin \{j, \lfloor j+1 \rfloor\}} \frac{\alpha_t}{1 - \alpha - \alpha_{\lfloor j+1 \rfloor}}\mathbf{p}_t + \frac{\alpha_j - \alpha}{1 - \alpha - \alpha_{\lfloor j+1 \rfloor}}\mathbf{p}_j.$$

Now we have that $\mathbf{p}_{\lfloor j+1 \rfloor} \in CM(P \setminus \{\mathbf{p}_{\lfloor j+1 \rfloor}\})$, again a contradiction with the first part of the lemma. Finally, if both terms are strictly positive, then

$$\frac{(\alpha - \alpha_j)\mathbf{p}_j + (1 - \alpha - \alpha_{\lfloor j+1 \rfloor})\mathbf{p}_{\lfloor j+1 \rfloor}}{1 - \alpha_j - \alpha_{\lfloor j+1 \rfloor}} > \sum_{t=1, t \notin \{j, \lfloor j+1 \rfloor\}}^{n} \frac{\alpha_j}{1 - \alpha_j - \alpha_{\lfloor j+1 \rfloor}}\mathbf{p}_t.$$

Denote the left hand side by $\mathbf{p}'''$, then the above inequality shows that $\mathbf{p}''' \in CM(P) \setminus \{\mathbf{p}_j\}$. Moreover, as $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1 \rfloor}$ and $1 = \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor}$, we have that $1 \geq \mathbf{p}'''\mathbf{q}_{\lfloor j+1 \rfloor}$, as $\mathbf{p}'''$ is a weighted average of both $\mathbf{p}_j$ and $\mathbf{p}_{\lfloor j+1 \rfloor}$. This shows that $H(\mathbf{q}_{\lfloor j+1 \rfloor})$ cuts the set $CM(P) \setminus \{\mathbf{p}_j\}$. Similar to before, we can thus use Lemma 1 (i.e. there exist a vertex $\mathbf{p}_v \in CM(P)$ distinct from $\mathbf{p}_j$ such that $1 \geq \mathbf{p}_v\mathbf{q}_{\lfloor j+1 \rfloor}$) to conclude that there must exists a strictly smaller subset of prices that implies a violation of SARP, which again gives us the desired contradiction. $\square$

We are now ready to give the proof of Proposition 2.

Proof. **Sufficiency:** Consider a set of prices $P = \{\mathbf{p}_t | t = 1, \ldots, n\}$ that satisfy the triangular configuration condition. If for all sets of bundles $\mathbf{q}_1, \ldots, \mathbf{q}_n$, $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ satisfies SARP, then evidently, WARP is also satisfied, so there is nothing left to prove. Therefore consider a set $\{\mathbf{q}_t | t = 1, \ldots, n\}$ of distinct bundles such that $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ violates SARP and assume, towards a contradiction, that it satisfies WARP. Note that we may consider the case where $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ contains no smaller subset that also violate SARP (since otherwise we could replace P by a smaller subset of prices). Next, let us renumber the observations such that the SARP violation is given by $\mathbf{q}_1 R \mathbf{q}_2 R \mathbf{q}_3 \ldots R \mathbf{q}_n R \mathbf{q}_1$ (i.e. $1 \geq \mathbf{p}_1\mathbf{q}_2, 1 \geq \mathbf{p}_2\mathbf{q}_3, \ldots, 1 \geq \mathbf{p}_n\mathbf{q}_1$).

Consider all three element subsets $\{\mathbf{p}_j, \mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}\}$. Given that P is a triangular configuration, we have that, for all j, there is a $\lambda \in [0,1]$ such that one of the following inequalities holds:

$$\mathbf{p}_j \leq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{1}$$

$$\mathbf{p}_{\lfloor j+1 \rfloor} \leq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{2}$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \leq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}, \tag{3}$$

$$\mathbf{p}_j \geq \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{4}$$

$$\mathbf{p}_{\lfloor j+1 \rfloor} \geq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}, \tag{5}$$

$$\mathbf{p}_{\lfloor j+2 \rfloor} \geq \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}. \tag{6}$$

If one of the latter inequalities (4)-(6) holds, then either $\mathbf{p}_j$, $\mathbf{p}_{\lfloor j+1 \rfloor}$ or $\mathbf{p}_{\lfloor j+2 \rfloor}$ is not a vertex of $CM(P)$, which contradicts Lemma 2. Given this, it must be that one of the inequalities (1)-(3) holds. Let us first show that (1) and (3) cannot hold.

Assume that (1) holds. Since $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n\}$ contains no subset that violates SARP, we know from Lemma 2 that the inequality cannot be strict. As such, it must be that the inequality holds with equality, i.e.

$$\mathbf{p}_j = \lambda \mathbf{p}_{\lfloor j+1 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}.$$

This implies that

$$\begin{aligned}
\mathbf{p}_j \mathbf{q}_{\lfloor j+2 \rfloor} &= \lambda \mathbf{p}_{\lfloor j+1 \rfloor} \mathbf{q}_{\lfloor j+2 \rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+2 \rfloor} \\
&\leq \lambda \mathbf{p}_{\lfloor j+1 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor} + (1-\lambda) \\
&\leq 1.
\end{aligned}$$

As such we obtain that the smaller data set $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots n; t \neq \lfloor j+1 \rfloor\}$ violates SARP, a contradiction.

Next assume that (3) holds. Again by Lemma 2, we have that the inequality cannot be strict and thus that

$$\mathbf{p}_{\lfloor j+2 \rfloor} = \lambda \mathbf{p}_j + (1-\lambda)\mathbf{p}_{\lfloor j+1 \rfloor}.$$

Observe that $1 < \mathbf{p}_{\lfloor j+2 \rfloor}\mathbf{q}_{\lfloor j+1 \rfloor}$, since otherwise $\{\mathbf{p}_{\lfloor j+1 \rfloor}, \mathbf{q}_{\lfloor j+1 \rfloor}, \mathbf{p}_{\lfloor j+2 \rfloor}, \mathbf{q}_{\lfloor j+2 \rfloor}\}$ violates WARP.

This implies

$$\begin{aligned}1 &< \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+1\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor} + (1-\lambda),\end{aligned}$$

which is equivalent to $1 < \mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor}$. This contradicts $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+1\rfloor}$.

We can thus conclude that for all j, (2) must hold. If for some j this inequality holds with an equality, then

$$\begin{aligned}1 &\geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+2\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda)\mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+2\rfloor}\\ &= \lambda\mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor} + (1-\lambda),\end{aligned}$$

which implies that $1 \geq \mathbf{p}_j\mathbf{q}_{\lfloor j+2\rfloor}$. But then $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n; t \neq \lfloor j+1\rfloor\}$ violates SARP, which gives a contradiction.

Given all this, it must be the case that there exist numbers $\lambda_j \in [0,1]$ such that,

$$\begin{aligned}&\lambda_1\mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3 > \mathbf{p}_2,\\ &\lambda_2\mathbf{p}_2 + (1-\lambda_2)\mathbf{p}_4 > \mathbf{p}_3,\\ &\ldots,\\ &\lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_{n-1})\mathbf{p}_1 > \mathbf{p}_n,\\ &\lambda_n\mathbf{p}_n + (1-\lambda_n)\mathbf{p}_2 > \mathbf{p}_1.\end{aligned}$$

Let us first show that for all j, $\lambda_j \notin \{0,1\}$. If $\lambda_j = 1$, we obtain that $\mathbf{p}_j > \mathbf{p}_{\lfloor j+1\rfloor}$. Then $1 = \mathbf{p}_j\mathbf{q}_j \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_j$ so we have that $\{\mathbf{p}_j, \mathbf{q}_j, \mathbf{p}_{\lfloor j+1\rfloor}, \mathbf{q}_{\lfloor j+1\rfloor}\}$ violates WARP. If $\lambda_j = 0$, we obtain that $\mathbf{p}_{\lfloor j+2\rfloor} > \mathbf{p}_{\lfloor j+1\rfloor}$. Then we have that $1 \geq \mathbf{p}_{\lfloor j+2\rfloor}\mathbf{q}_{\lfloor j+3\rfloor} \geq \mathbf{p}_{\lfloor j+1\rfloor}\mathbf{q}_{\lfloor j+3\rfloor}$. This implies that the smaller data set $\{(\mathbf{p}_t, \mathbf{q}_t) | t = 1, \ldots, n; t \neq \lfloor j+2\rfloor\}$ violates SARP.

Now, let us show by induction on n that above system of inequalities with $\lambda_j \in]0,1[$ can not have a solution for the λ_j.

If $n = 3$, we obtain the system,

$$\begin{aligned}&\lambda_1\mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3 > \mathbf{p}_2,\\ &\lambda_2\mathbf{p}_2 + (1-\lambda_2)\mathbf{p}_1 > \mathbf{p}_3,\\ &\lambda_3\mathbf{p}_3 + (1-\lambda_3)\mathbf{p}_2 > \mathbf{p}_1.\end{aligned}$$

This gives

$$
\begin{aligned}
\mathbf{p}_2 &< \lambda_1 \mathbf{p}_1 + (1-\lambda_1)\mathbf{p}_3,\\
\mathbf{p}_2 &> -\frac{(1-\lambda_2)}{\lambda_2}\mathbf{p}_1 + \frac{1}{\lambda_2}\mathbf{p}_3,\\
\mathbf{p}_2 &> -\frac{\lambda_3}{1-\lambda_3}\mathbf{p}_3 + \frac{1}{1-\lambda_3}\mathbf{p}_1.
\end{aligned}
$$

Combining these inequalities leads to

$$
\begin{aligned}
0 &< \left(\lambda_1 + \frac{(1-\lambda_2)}{\lambda_2}\right)\mathbf{p}_1 + \left((1-\lambda_1) - \frac{1}{\lambda_2}\right)\mathbf{p}_3,\\
0 &< \left(\lambda_1 - \frac{1}{1-\lambda_3}\right)\mathbf{p}_1 + \left((1-\lambda_1) + \frac{\lambda_3}{1-\lambda_3}\right)\mathbf{p}_3,
\end{aligned}
$$

which gives the contradiction,

$$\mathbf{p}_1 > \mathbf{p}_3 \text{ and } \mathbf{p}_1 < \mathbf{p}_3.$$

For the induction step, assume that there is no solution for any set of n prices and consider a system of inequalities with $n+1$ prices. The inequalities involving $\mathbf{p}_{n+1}$ are given by

$$
\begin{aligned}
&\lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_{n-1})\mathbf{p}_{n+1} > \mathbf{p}_n,\\
&\lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 > \mathbf{p}_{n+1},\\
&\lambda_{n+1}\mathbf{p}_{n+1} + (1-\lambda_{n+1})\mathbf{p}_2 > \mathbf{p}_1.
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
\mathbf{p}_{n+1} &> \frac{1}{1-\lambda_{n-1}}\mathbf{p}_n - \frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1},\\
\mathbf{p}_{n+1} &< \lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1,\\
\mathbf{p}_{n+1} &> -\frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 + \frac{1}{\lambda_{n+1}}\mathbf{p}_1
\end{aligned}
$$

Combining these inequalities leads to

$$
\begin{aligned}
&\lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 > \frac{1}{1-\lambda_{n-1}}\mathbf{p}_n - \frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1},\\
&\lambda_n \mathbf{p}_n + (1-\lambda_n)\mathbf{p}_1 > -\frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 + \frac{1}{\lambda_{n+1}}\mathbf{p}_1\\
\iff &\frac{\lambda_{n-1}}{1-\lambda_{n-1}}\mathbf{p}_{n-1} + (1-\lambda_n)\mathbf{p}_1 > \left(\frac{1}{1-\lambda_{n-1}} - \lambda_n\right)\mathbf{p}_n,\\
&\lambda_n \mathbf{p}_n + \frac{1-\lambda_{n+1}}{\lambda_{n+1}}\mathbf{p}_2 > \left(\frac{1}{\lambda_{n+1}} - (1-\lambda_n)\right)\mathbf{p}_1\\
\iff &\lambda_{n-1}\mathbf{p}_{n-1} + (1-\lambda_n)(1-\lambda_{n-1})\mathbf{p}_1 > (1-\lambda_n(1-\lambda_{n-1}))\,\mathbf{p}_n,\\
&\lambda_n\lambda_{n+1}\mathbf{p}_n + (1-\lambda_{n+1})\mathbf{p}_2 > (1-(1-\lambda_n)\lambda_{n+1})\,\mathbf{p}_1\\
\iff &\frac{\lambda_{n-1}}{1-\lambda_n(1-\lambda_{n-1})}\mathbf{p}_{n-1} + \frac{(1-\lambda_n)(1-\lambda_{n-1})}{1-\lambda_n(1-\lambda_{n-1})}\mathbf{p}_1 > \mathbf{p}_n,\\
&\frac{\lambda_n\lambda_{n+1}}{1-(1-\lambda_n)\lambda_{n+1}}\mathbf{p}_n + \frac{(1-\lambda_{n+1})}{1-(1-\lambda_n)\lambda_{n+1}}\mathbf{p}_2 > \mathbf{p}_1.
\end{aligned}
$$

Let us denote $\lambda'_{n-1} = \frac{\lambda_{n-1}}{1-\lambda_n(1-\lambda_{n-1})}$ and $\lambda'_n = \frac{\lambda_n\lambda_{n+1}}{1-(1-\lambda_n)\lambda_{n+1}}$. It is easily verified that $\lambda'_{n-1}, \lambda'_n \in]0,1[$. Substitution then gives

$$\lambda'_{n-1}\mathbf{p}_{n-1} + (1-\lambda'_{n-1})\mathbf{p}_1 > \mathbf{p}_n \text{ and } \lambda'_n\mathbf{p}_n + (1-\lambda'_n)\mathbf{p}_2 > \mathbf{p}_1.$$

Thus, we effectively substituted the last three inequalities of the system with $n+1$ prices by the last two inequalities for the system with only n prices. From the induction hypothesis, we know that this system has no feasible solution. This infeasibility finishes the sufficiency part of our proof, since we can conclude that the triangular configuration implies WARP-reducibility.

Necessity: To show the reverse, let us consider a set of prices P that is not a triangular configuration. In particular, let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ be three distinct price vectors such that none of the vector inequalities is satisfied. First of all, as the triangular configuration is not satisfied, it must be that the three prices form the vertices of the convex set $CM(\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\})$.

Consider the convex sets $CM(\{\mathbf{p}_1, \mathbf{p}_2\})$ and $C(\{\mathbf{p}_1, \mathbf{p}_3\})$. Then,

$$CM(\{\mathbf{p}_1, \mathbf{p}_2\}) \cap (C(\{\mathbf{p}_1, \mathbf{p}_3\}) \setminus \{\mathbf{p}_1, \mathbf{p}_3\}) = \emptyset.$$

In order to see this, assume, towards a contradiction, that there exists a vector $\mathbf{p}$ such that $\mathbf{p} \geq \lambda\mathbf{p}_1 + (1-\lambda)\mathbf{p}_2$ and $\mathbf{p} = \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_3$ (with $\alpha \in]0,1[$). Then, substitution gives

$$(\alpha - \lambda)\mathbf{p}_1 + (1-\alpha)\mathbf{p}_3 \geq (1-\lambda)\mathbf{p}_2.$$

This implies that $\lambda \neq 1$, since otherwise $\mathbf{p}_3 \geq \mathbf{p}_1$ which contradicts with $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ not being a triangular configuration. If $\alpha \geq \lambda$, then

$$\frac{(\alpha-\lambda)}{1-\lambda}\mathbf{p}_1 + \frac{(1-\alpha)}{1-\lambda}\mathbf{p}_3 \geq \mathbf{p}_2.$$

This shows that a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$ is larger than $\mathbf{p}_2$, which again implies that the prices form a triangular configuration. On the other hand, if $\lambda > \alpha$, then

$$\mathbf{p}_3 \geq \frac{\lambda-\alpha}{1-\alpha}\mathbf{p}_1 + \frac{1-\lambda}{1-\alpha}\mathbf{p}_2.$$

This shows that $\mathbf{p}_3$ is larger than the convex combination of $\mathbf{p}_1$ and $\mathbf{p}_2$, again showing that the prices form a triangular configuration. This proves our conjecture.

Therefore, from the supporting hyperplane theorem, we know that there exists a hyperplane $H(\mathbf{q}_1)$ with $\mathbf{p}_1\mathbf{q}_1 = 1$, $1 < \mathbf{p}_2\mathbf{q}_1$ and $1 \geq \mathbf{p}_3\mathbf{q}_1$. We can of course repeat this reasoning, by exchanging the indices, in order to show that there also exist a $\mathbf{q}_2$ and a $\mathbf{q}_3$ satisfying similar constraints. All this implies that there exist $\mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{q}_3$ such that

$$\begin{aligned}
&\mathbf{p}_1\mathbf{q}_1 = 1, \mathbf{p}_2\mathbf{q}_2 = 1, \mathbf{p}_3\mathbf{q}_3 = 1,\\
&1 \geq \mathbf{p}_1\mathbf{q}_2, 1 \geq \mathbf{p}_2\mathbf{q}_3, 1 \geq \mathbf{p}_3\mathbf{q}_1,\\
&1 < \mathbf{p}_1\mathbf{q}_3, 1 < \mathbf{p}_2\mathbf{q}_1, 1 < \mathbf{p}_3\mathbf{q}_2.
\end{aligned}$$

This implies a cycle of length 3 that violates SARP, while there is no WARP violation. □

B Factorization procedure

Start from a dataset with n observations and m goods. Consider a number $k < m$ and denote by $\mathcal{P}$ the $n \times m$ matrix where all row vectors $\mathbf{p}_t$ are stacked above each other. A non-negative matrix factorization of $\mathcal{P}$ consists of a non-negative matrix W of dimension $n \times k$ and a non-negative matrix M of dimension $k \times m$ such that

$$\underset{(n\times m)}{\mathcal{P}} \approx \underset{(n\times k)}{W}\underset{(k\times m)}{M}.$$

Usually, the approximation is close in the sense that it minimizes the sum of squares $\sum_{t=1}^{n}\sum_{l=1}^{m}\left(p_{t,l} - \sum_{i=1}^{k} W_{t,i}M_{i,l}\right)^2$. Let $\mathbf{w}_t$ be the t-th row vector of W. Then, if the factorization is close to exact, we can write $\mathbf{p}_t \approx \mathbf{w}_t M$. We also have that, for any vectors $\mathbf{q}_t$ and $\mathbf{q}_v$, $\mathbf{q}_t R \mathbf{q}_v$ if and only if

$$\mathbf{p}_t\mathbf{q}_t \geq \mathbf{p}_t\mathbf{q}_v \iff 1 \geq \mathbf{w}_t M \mathbf{q}_v \iff 1 \geq \mathbf{w}_t\mathbf{z}_v,$$

where $\mathbf{z}_v = M\mathbf{q}_v$. The vector $\mathbf{w}_t$ can be interpreted as a k-dimensional price vector and the vector $\mathbf{z}_v$ as a k-dimensional quantity vector. As such, if the factorization is exact, we obtain that $\mathbf{q}_t R \mathbf{q}_v$ for the dataset $\{(\mathbf{p}_t, \mathbf{q}_t)|t = 1, \ldots, n\}$ if and only if $\mathbf{z}_t R \mathbf{z}_v$ for the dataset $\{(\mathbf{w}_t, \mathbf{z}_t)|t = 1, \ldots, n\}$. By factorizing the matrix $\mathcal{P}$, we therefore reduced the number of goods from m to k. In our application, we use the factorization algorithm of

Kim and Park (2008). We consider $k = 3, 5, 10$ and 15. For these alternative k-values, we verify the triangular conditions on the normalized prices $\mathbf{w}_t/(\mathbf{w}_t\mathbf{z}_t)$. We remark that, if the factorization is exact, then $\mathbf{p}_t = \mathbf{w}_t M$ and thus $\mathbf{w}_t\mathbf{z}_t = 1$, so that this normalization becomes redundant.

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